

Thermal conductivity in 3D NJL model under external magnetic field

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Abstract. The thermal conductivity of the (2+1)-dimensional NJL model in the presence of a constant magnetic field is calculated in the mean-field approximation and its different asymptotic regimes are analyzed. Taking into account the dynamical generation of a fermion mass due to the magnetic catalysis phenomenon, it is shown that for certain relations among the theory's parameters (particle width, temperature and magnetic field), the profile of the thermal conductivity *versus* the applied field exhibits kink- and plateau-like behaviors. We point out possible applications to planar condensed matter.

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1 Introduction

It is now well established, from the study of many relativistic theories of massless fermions in the presence of an external magnetic field, that a magnetic field can be a strong catalyst for chiral symmetry breaking with the consequent generation of a fermion dynamical mass even at the weakest attractive interaction among fermions [1]. This magnetic catalysis (MC) of chiral symmetry breaking has proven to be universal, its main features being independent of the model under consideration (see Refs. [1–8] for various aspects of this phenomenon). The universality character of the magnetic catalysis has motivated many recent works [5–13] aimed to apply it to diverse areas of quantum physics. The essence of the MC effect lies in the dimensional reduction of the fermion pairing dynamics due to the confinement of these particles to their lowest Landau level (LLL), when the pairing energy is much less than the Landau gap \sqrt{B} (B is the magnitude of the magnetic field induction). Under these circumstances, any attraction between fermions, whenever small it might be, is strengthened by the effective dimensional reduction in the presence of the magnetic field, and therefore, a condensate of fermion-antifermion is formed with the subsequent generation of a fermion mass. The lowest LL plays in this case a role similar to that of the Fermi surface in BCS superconductivity [1].

It is known that several quasiplanar systems have low-energy excitation spectrum of quasiparticles (QP), characterized by a linear dispersion, around the Fermi surface consisting of isolated points. The dynamics of these QP can be described by a “relativistic” quantum field theory of massless fermions. When such Dirac-like QP are electrically charged, they can couple to an externally applied magnetic field which can catalyze the condensation of QP-antiQP pairs. Then, one would expect the realization of MC in such a kind of condensed matter systems. As a matter of fact, the MC was suggested as the possible explanation [8–12] for the profile of the thermal conductivity in an applied magnetic field observed in recent experiments in planar high- T_c cuprates [14–17]. The MC has also been proposed [18] as the source of the semimetal-insulator phase transition observed in the so-called highly oriented pyrolytic graphites (HOPG) [19] in the presence of a magnetic field.

Given that the heat transport is a convenient probe to understand many basic properties of quasi-planar condensed matter systems, as gap structure, QP density, scattering rate, etc, the study of heat transport in a quasi-planar system subjected to MC may turn out physically revealing. It is the goal of the present paper to study how the MC affects the thermal transfer properties of a (2+1)-dimensional fermion system under an applied constant magnetic field. In particular, we calculate the thermal conductivity of a system of QP described by a (2+1)-dimensional Nambu-Jona-Lasinio (NJL) model

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which exhibits a mass (gap) generation for fermions in the presence of a magnetic field, and discuss possible applications to planar condensed matter systems. To obtain the results here reported, we used the same approximation of a constant magnetic field that was already explored in references [9, 10]. However, our calculations deviate considerably from what was done in these papers. Not only we take into account the contribution of all Landau levels, but the definition we use for the heat current itself is different. In our formulation, when the gap induced by the magnetic field is opened, the thermal conductivity exhibits a new term proportional to σ^2 (σ is the gap). Near the phase transition point, the gap behaves like $\sigma \sim \sqrt{B - B_c}$ in the mean-field approximation. Hence, the term proportional to σ^2 yields a positive contribution in the slope of the thermal conductivity, leading to a jump in the slope of $\kappa(B)$ at $B = B_c$ (kink-like behavior). Notice that the magnetic catalysis is mainly responsible for the kink effect, being the critical behavior of the gap an essential factor for this result.

We underline that to obtain such a kink-like behavior, it is crucial to go beyond the LLL contribution in the calculation of the thermal conductivity, since, as we show, the heat transfer takes place due to transitions between neighboring LLs. Physically, this is easy to understand keeping in mind that the spatial momentum of the QP lies in the system plane. In the presence of a magnetic field perpendicularly applied to the two-dimensional sample, the QP spatial momentum is purely transverse and hence quantized into discrete Landau levels. Therefore, the transfer of kinetic energy can only occur by means of transitions between Landau levels.

Even though our results were obtained by using a particular model, we point out that the main outcome of the present work is of a more general and theoretical character, as we show that the MC phenomenon can be responsible for a kink-like effect in the thermal conductivity of a whole class of (2+1)-dimensional relativistic fermion systems. That is, we show that the kink effect is essentially model independent, since it is determined by the critical behavior of the dynamically generated mass near the phase transition point. This fact makes the basic outcome of our investigation relevant beyond the particular model under consideration, linking it to the universality class of theories with such a critical behavior. In connection with this we conjecture that, since the HOPG may be described [18] by a model that belongs to the same universality class as that of the model used here, the HOPG thermal conductivity in the presence of a magnetic field should display similar kink-like behavior.

The plan of the paper is as follows. In Section 2 we derive the expression for the thermal conductivity in the (2+1)-dimensional NJL model in the presence of a constant magnetic field and analytically study its different asymptotics, underlying the possible application of each result. In Section 3 we obtain the thermal conductivity *vs.* magnetic field profile using numerical calculations. In the reported graph, the change of slope in the thermal conductivity profile is shown to occur at the critical mag-

netic field where the fermion dynamical mass is generated at the given temperature. The conclusions and discussion of potential applications of our results are presented in Section 4. In Appendix A we derive the critical curve in the $B - T$ plane which separates the symmetric and the symmetry-broken phases in the (2+1)-dimensional NJL model, and study the scaling of the mass near the critical curve. A derivation of the Kubo formula in the framework of Matsubara formalism is given in Appendix B.

2 Thermal conductivity in the (2+1)-D NJL model in the presence of a constant magnetic field

2.1 General results

We start from the (2+1)-dimensional NJL Lagrangian density in an external magnetic field

$$\mathcal{L} = \frac{1}{2} [\bar{\psi}_i, i\gamma^\mu D_\mu \psi_i] + \frac{g}{2N} (\bar{\psi}_i \psi_i)^2, \quad (1)$$

where $D_\mu = \partial_\mu - ieA_\mu^{ext}$ is the covariant derivative and the vector potential for the external magnetic field is taken in the symmetric gauge

$$A_\mu^{ext} = \left(0, -\frac{B}{2}x_2, \frac{B}{2}x_1 \right), \quad (2)$$

and we assume that the fermions carry an additional flavor index $i = 1, \dots, N$. In the case of d -wave superconductors the number N is equal to $2n_{\text{CuO}}$, with n_{CuO} denoting the number of CuO planes per unit cell ($N = 2$ for planar d -wave superconductors). For a single sheet of a HOPG $N = 2$ too. The Dirac γ -matrices are taken in the reducible four-component representation.

In the absence of the bare mass term $m\bar{\psi}\psi$, the Lagrangian density (1) is invariant under discrete chiral symmetry

$$\psi \rightarrow \gamma_5 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi} \gamma_5, \quad (3)$$

which forbids the fermion mass generation in perturbation theory. The appearance of the mass (energy gap) is due to the spontaneous breaking of the above discrete symmetry that leads to a neutral condensate of fermion-antifermion pairs. In condensed matter physics this could correspond to the condensation of excitons (electron-hole bound states).

Introducing the composite field $\sigma = -g(\bar{\psi}_i \psi_i)/N$, the Lagrangian (1) can be written in the form

$$\mathcal{L} = \frac{1}{2} [\bar{\psi}_i, i\gamma^\mu D_\mu \psi_i] - \sigma \bar{\psi}_i \psi_i - \frac{N\sigma^2}{2g}. \quad (4)$$

One readily verifies the equivalence of the Lagrangians (1) and (4) by making use of the Euler-Lagrange equations (or performing the integration over the field σ in the functional integral). The field σ has no dynamics at the tree level, however, it acquires a kinetic term due to fermion loops. The vacuum expectation value of σ gives a dynamical mass (gap) to fermions. The effective action for the composite field σ can be obtained by integrating over fermions in the path integral (see Appendix A).

It is well known that in the absence of a magnetic field and at zero temperature the mass generation occurs only if the coupling constant exceeds some critical value [20]. This can be seen from the stationary equation for the effective potential corresponding to the Lagrangian density (1), which in leading order of $1/N$ -expansion is given by

$$\frac{\partial V(\sigma)}{\partial \sigma} = \sigma \left[\sigma - \frac{\Lambda}{\sqrt{\pi}} + \frac{\Lambda}{\bar{g}} \right] = 0 \quad (5)$$

(it follows from Eq. (A.4) in the limit $B = T = 0$). In (5) we introduced the dimensionless coupling constant $\bar{g} = \Lambda g / \pi$ and Λ is the ultraviolet cutoff parameter. The large- N expansion, originally borrowed from studies of dynamical symmetry breaking in a particle physics context [20], is widely used in condensed matter problems [18, 21, 22] as a feasible technique that gives reliable qualitative results at leading order even in cases where the physical N corresponds to $N = 2$.

From equation (5) it is easy to see that there exists a critical value for the coupling constant $\bar{g}_c = \sqrt{\pi}$ such that, if $\bar{g} < \bar{g}_c$ equation (5) has only the trivial solution ($\sigma = 0$), while for the strong coupling limit $\bar{g} > \bar{g}_c$ a non-trivial solution ($\bar{\sigma} = \Lambda(\bar{g} - \sqrt{\pi}) / \bar{g}\sqrt{\pi}$) is reached which leads to the generation of a fermion mass. Hence, the critical coupling \bar{g}_c separates two phases: the weakly coupled massless phase at $\bar{g} < \bar{g}_c$ and the strongly coupled massive one ($\bar{g} > \bar{g}_c$). An applied magnetic field changes the situation dramatically, so that the mass generation now takes place at all $\bar{g} > 0$ [1–4], hence the name magnetic catalysis. At finite T and $B \neq 0$ there is a critical curve in the $B - T$ plane separating the symmetric and the symmetry broken phases (the derivation and analysis of the critical curve in the $B - T$ plane are given in Appendix A).

To derive an expression for the static thermal conductivity in an isotropic system we follow the familiar linear response method and apply Kubo's formula [23]

$$\kappa = -\frac{1}{TV} \text{Im} \int_0^\infty dt \int d^2x_1 d^2x_2 \langle u_i(x_1, 0) u_i(x_2, t) \rangle, \quad (6)$$

where V is the volume of the system, $T = 1/\beta$ is the temperature, and $u_i(x, t)$ is the heat-current density operator. The brackets denote averaging in the canonical ensemble with the density matrix $\rho = e^{-\beta H} / Z$, $Z = \text{Tr} e^{-\beta H}$.

Physically the thermal conductivity κ appears as a coefficient in the equation relating the heat current to the temperature gradient

$$\mathbf{u} = -\kappa \nabla T \quad (7)$$

under the condition of absence of particle flow. If we neglect the chemical potential the heat density coincides with the energy density, hence the quantity that satisfies the continuity equation

$$\dot{\epsilon}(x) + \nabla \cdot \mathbf{u}(x) = 0 \quad (8)$$

can be interpreted as the heat current density. Equation (8) defines \mathbf{u} to within a divergenceless vector, which is sufficient for calculating the conductivity. The vector \mathbf{u} is obtained automatically from the Lagrangian density (1) as

$$u_i = \frac{\partial \mathcal{L}}{\partial(\partial^i \psi)} \dot{\psi} + \bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial^i \bar{\psi})} = \frac{i}{2} (\bar{\psi} \gamma_i \partial_0 \psi - \partial_0 \bar{\psi} \gamma_i \psi). \quad (9)$$

When using equations of motion the last expression can be represented in the form

$$u_i = \frac{i}{2} (\bar{\psi} \gamma^0 D_i \psi - \overline{D_i \psi} \gamma^0 \psi). \quad (10)$$

At this point it is useful to underline that our definition of the heat current does not coincide with the one used in references [9, 10], $P_i(x) = \bar{\psi} \gamma^0 \partial_i \psi - \partial_i \bar{\psi} \gamma^0 \psi$. Operator $P_i(x)$ cannot be obtained from equation (9) by using the equations of motion, unless the external field is zero, so it does not lead to the correct thermal conductivity in the presence of a magnetic field. Note also, that our quantity u_i is explicitly gauge invariant in contrast to P_i .

The correlator of the heat currents (or polarization function) in (6) is evaluated in the following way (see Appendix B for details on the formalism). First, it is computed in the Matsubara finite temperature formalism replacing the time t by the imaginary time τ ($t = -i\tau$):

$$\begin{aligned} \Pi(i\Omega_m) &= \frac{1}{V} \int_0^\beta d\tau e^{i\Omega_m \tau} \langle T_\tau U_i(\tau) U_i(0) \rangle, \\ U_i(\tau) &= \int d^2x u_i(x, \tau), \quad \Omega_m = \frac{2\pi m}{\beta}. \end{aligned} \quad (11)$$

The thermal conductivity is then given by the discontinuity of the retarded function $\Pi^R(\Omega)$, which is obtained by analytic continuation from imaginary discrete frequencies $\Pi^R(\Omega) = \Pi(i\Omega_m \rightarrow \Omega + i\epsilon)$:

$$\kappa = \frac{1}{4T_i} \lim_{\Omega \rightarrow 0} \frac{1}{\Omega} [\Pi^R(\Omega + i\epsilon) - \Pi^A(\Omega - i\epsilon)]. \quad (12)$$

Neglecting vertex corrections¹, the calculation of the thermal conductivity reduces to the evaluation of the bubble diagram

$$\begin{aligned} \Pi(i\Omega_m) = & -T \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} \\ & \times \text{tr} \left[\gamma^i i\omega_n S(i\omega_n, \mathbf{k}) \gamma^i (i\omega_n + i\Omega_m) S(i\omega_n + i\Omega_m, \mathbf{k}) \right], \end{aligned} \quad (13)$$

where $S(\omega, k)$ is the Fourier transform of the translation invariant part $\tilde{S}(x-y)$ of the fermion propagator in an external magnetic field:

$$S(x-y) = \exp \left(ie \int_y^x A_\lambda^{ext} dz^\lambda \right) \tilde{S}(x-y). \quad (14)$$

Note, that the translation non-invariant phase of the fermion Green's function cancels in the computation of Π . Defining a spectral representation for $S(i\omega_n, k)$, we can write

$$S(i\omega_n, \mathbf{k}) = \int_{-\infty}^{\infty} \frac{d\omega A(\omega, \mathbf{k})}{i\omega_n - \omega}. \quad (15)$$

The spectral representation allows one to make analytic continuation and find the retarded, S^R , and advanced, S^A , Green functions according to the rule $S^R(\omega + i\epsilon, \mathbf{k}) = S(i\omega_n = \omega + i\epsilon, \mathbf{k})$ and $S^A(\omega - i\epsilon, \mathbf{k}) = S(i\omega_n = \omega - i\epsilon, \mathbf{k})$. The spectral function $A(\omega, \mathbf{k})$ is given by

$$A(\omega, \mathbf{k}) = \frac{1}{2\pi i} [S^A(\omega - i\epsilon, \mathbf{k}) - S^R(\omega + i\epsilon, \mathbf{k})]. \quad (16)$$

Plugging the spectral representation (15) into equation (13) the sum over Matsubara frequencies can be performed². After this has been done, we can continue the external frequencies to the real axis to get $\Pi^R(\Omega)$. Finally, we arrive at the following expression for the thermal conductivity

$$\kappa = \frac{1}{32\pi T^2} \int_{-\infty}^{\infty} \frac{d\omega \omega^2}{\cosh^2 \frac{\omega}{2T}} \int d^2k \text{tr} [\gamma^i A(\omega, \mathbf{k}) \gamma^i A(\omega, \mathbf{k})]. \quad (17)$$

In order to compute κ we have to specify now the fermion propagator. Since the fermion mass is generated in the (2+1)-dimensional NJL model in a magnetic field already

¹ It has been argued that for small impurity densities the thermal conductivity, unlike the electric conductivity, is unaffected by vertex corrections [24].

² There is a subtle point here since the sum over frequencies appears to be divergent. However, as was shown in [23], this divergence results from an improper treatment of time derivatives inside the time-ordered product of currents in (11). This divergence disappears when the problem is treated more carefully. The prescription is simply to ignore it.

at weak coupling, we take the standard expression for the massive fermion propagator in a magnetic field, decomposed over the Landau level poles [2,25]

$$S(\omega, \mathbf{k}) = e^{-\frac{\mathbf{k}^2}{eB}} \sum_{n=0}^{\infty} (-1)^n \frac{D_n(\omega, \mathbf{k})}{\omega^2 - \sigma^2 - 2eBn}, \quad (18)$$

where

$$\begin{aligned} D_n(\omega, k) = & 2(\omega\gamma^0 + \sigma) \left[P_- L_n \left(\frac{2\mathbf{k}^2}{eB} \right) - P_+ L_{n-1} \left(\frac{2\mathbf{k}^2}{eB} \right) \right] \\ & + 4\mathbf{k}\gamma L_{n-1}^1 \left(\frac{2\mathbf{k}^2}{eB} \right) \end{aligned} \quad (19)$$

with $P_\pm = (1 \pm i\gamma^1\gamma^2)/2$ being projectors and L_n, L_n^1 Laguerre's polynomials ($L_{-1}^1 \equiv 0$). Here σ is the fermion dynamical mass obtained from the finite temperature gap equation in a constant magnetic field (see Appendix A).

The spectral function according to (16) is found to be

$$\begin{aligned} A(\omega, \mathbf{k}) = & e^{-\frac{\mathbf{k}^2}{eB}} \frac{\Gamma}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{M_n} \\ & \times \left[\frac{(\gamma^0 M_n + \sigma) f_1(\mathbf{k}) + f_2(\mathbf{k})}{(\omega - M_n)^2 + \Gamma^2} \right. \\ & \left. + \frac{(\gamma^0 M_n - \sigma) f_1(\mathbf{k}) - f_2(\mathbf{k})}{(\omega + M_n)^2 + \Gamma^2} \right], \end{aligned} \quad (20)$$

where $M_n = \sqrt{\sigma^2 + 2eBn}$ and

$$\begin{aligned} f_1(\mathbf{k}) = & 2 \left[P_- L_n \left(\frac{2\mathbf{k}^2}{eB} \right) - P_+ L_{n-1} \left(\frac{2\mathbf{k}^2}{eB} \right) \right], \\ f_2(\mathbf{k}) = & 4\mathbf{k}\gamma L_{n-1}^1 \left(\frac{2\mathbf{k}^2}{eB} \right). \end{aligned} \quad (21)$$

Here, we introduced the width Γ of the quasiparticles, which is due to interaction processes, in particular, scattering on impurities, having replaced ϵ in (16) by a finite Γ . In general, the scattering rate Γ , which is defined through the fermion self-energy, $\Gamma(\omega) = -\text{Im}\Sigma^R(\omega)$, is a frequency-dependent quantity (as well as temperature and field dependent). It must be determined, like the dynamical mass, self-consistently from the Schwinger-Dyson equations. At low temperatures we are interested in its value at $\omega = 0$, so we will consider it as a phenomenological parameter.

In the absence of a magnetic field ($B = 0$) it can be shown that the spectral function (20) reduces to

$$\begin{aligned} A(\omega, \mathbf{k}) = & \frac{\Gamma}{2\pi E} \left[\frac{\gamma^0 E - \mathbf{k}\gamma + \sigma}{(\omega - E)^2 + \Gamma^2} + \frac{\gamma^0 E + \mathbf{k}\gamma - \sigma}{(\omega + E)^2 + \Gamma^2} \right], \\ E = & \sqrt{\mathbf{k}^2 + \sigma^2}, \end{aligned} \quad (22)$$

hence, the retarded fermion Green's function is

$$S^R(\omega, \mathbf{k}) = \frac{\gamma_0(\omega + i\Gamma) - \mathbf{k}\boldsymbol{\gamma} + \sigma}{(\omega + i\Gamma)^2 - \mathbf{k}^2 - \sigma^2}. \quad (23)$$

Straightforward calculation of the trace in (17), with $A(\omega, \mathbf{k})$ from equations (20, 21) gives

$$\text{tr} [\gamma^i A(\omega, \mathbf{k}) \gamma^i A(\omega, \mathbf{k})] =$$

$$\begin{aligned} & \frac{16\Gamma^2 N}{\pi^2} e^{-2\mathbf{k}^2/eB} \sum_{n,m=0}^{\infty} (-1)^{m+n+1} \\ & \times \frac{(\omega^2 + M_n^2 + \Gamma^2)(\omega^2 + M_m^2 + \Gamma^2) - 4\omega^2\sigma^2}{[(\omega^2 + M_n^2 + \Gamma^2)^2 - 4\omega^2 M_n^2]} \\ & \times \frac{1}{[(\omega^2 + M_m^2 + \Gamma^2)^2 - 4\omega^2 M_m^2]} \\ & \times \left(L_n \left(\frac{2\mathbf{k}^2}{eB} \right) L_{m-1} \left(\frac{2\mathbf{k}^2}{eB} \right) + L_{n-1} \left(\frac{2\mathbf{k}^2}{eB} \right) L_m \left(\frac{2\mathbf{k}^2}{eB} \right) \right). \end{aligned} \quad (24)$$

Performing now the integration over momenta in equation (17) produces Kronecker's delta symbols $\delta_{n,m-1}$ + $\delta_{m,n-1}$ due to the orthogonality of Laguerre's polynomials, thus we get

$$\begin{aligned} \kappa &= \frac{eB\Gamma^2 N}{\pi^2 T^2} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\omega\omega^2}{\cosh^2 \frac{\omega}{2T}} \\ & \times \frac{(\omega^2 + M_n^2 + \Gamma^2)(\omega^2 + M_{n+1}^2 + \Gamma^2) - 4\omega^2\sigma^2}{[(\omega^2 + M_n^2 + \Gamma^2)^2 - 4\omega^2 M_n^2]} \\ & \times \frac{1}{[(\omega^2 + M_{n+1}^2 + \Gamma^2)^2 - 4\omega^2 M_{n+1}^2]}. \end{aligned} \quad (25)$$

Note that the factor eB in front of the right hand side of (25) originated from integrating over transverse momenta and gives the degeneracy of Landau levels (more exactly, the degeneracy is $NeB/2\pi$ for the lowest LL, $n = 0$, and NeB/π for levels with $n \geq 1$). We stress the important result that because of the appearance of the mentioned Kronecker deltas only transitions between neighboring Landau levels contribute into the heat transfer. Had we restrict ourselves to the LLL as in [10], we would have gotten zero result for κ .

Further summation over n in equation (25) can be performed expanding the integrand in terms of partial fractions. The resulting sums are expressed through digamma functions by means of the formula

$$\begin{aligned} \sum_{n=0}^{\infty} \left[\frac{A}{n+a} + \frac{B}{n+b} + \frac{C}{n+c} + \frac{D}{n+d} \right] &= \\ & - [A\psi(a) + B\psi(b) + C\psi(c) + D\psi(d)], \end{aligned} \quad (26)$$

where for convergence $A + B + C + D = 0$.

After some algebraic manipulations the final expression for κ is written as follows

$$\begin{aligned} \kappa &= \frac{N\Gamma^2}{2\pi^2 T^2} \int_0^{\infty} \frac{d\omega\omega^2}{\cosh^2 \frac{\omega}{2T}} \frac{1}{(eB)^2 + (2\omega\Gamma)^2} \\ & \times \left\{ 2\omega^2 + \frac{(\omega^2 + \sigma^2 + \Gamma^2)(eB)^2 - 2\omega^2(\omega^2 - \sigma^2 + \Gamma^2)eB}{(\omega^2 - \sigma^2 - \Gamma^2)^2 + 4\omega^2\Gamma^2} \right. \\ & \quad \left. - \frac{\omega(\omega^2 - \sigma^2 + \Gamma^2)}{\Gamma} \text{Im}\psi \left(\frac{\sigma^2 + \Gamma^2 - \omega^2 - 2i\omega\Gamma}{2eB} \right) \right\}. \end{aligned} \quad (27)$$

This formula is the main result of our paper. Note that it is independent of the particular model used to describe the QP interactions unless we specify the dependence of the dynamical mass on Γ, T, eB for a concrete model.

Another representation of equation (27) which is particularly convenient for studying the small width limit $\Gamma \ll T, \sqrt{eB}$ is

$$\begin{aligned} \kappa &= \frac{N\Gamma^2}{4\pi^2 T^2} \int_{-\infty}^{\infty} \frac{d\omega\omega^2}{\cosh^2 \frac{\omega}{2T}} \frac{1}{(eB)^2 + (2\omega\Gamma)^2} \\ & \times \left\{ 2\omega^2 + \frac{(eB)^2}{2} + \frac{eB\omega(\omega + \sigma)}{(\omega + \sigma)^2 + \Gamma^2} + \frac{(eB)^2}{2} + \frac{eB\omega(\omega - \sigma)}{(\omega - \sigma)^2 + \Gamma^2} \right. \\ & \quad \left. + eB\omega \sum_{n=1}^{\infty} \frac{1}{M_n} \left[\frac{\sigma^2 + M_n^2 + 2\omega M_n}{(\omega + M_n)^2 + \Gamma^2} + \frac{2\omega M_n - \sigma^2 - M_n^2}{(\omega - M_n)^2 + \Gamma^2} \right] \right\}. \end{aligned} \quad (28)$$

It is obtained from (27) if one takes the series representation of the ψ -function in the integrand of (27) and writes the expression in the curved brackets in fractions of $1/(\Gamma^2 + x^2)$.

We are now in the position to study different asymptotic regimes defined by different relations among the dimensional parameters σ, Γ, T, B .

2.2 Zero magnetic field

First, we consider the limit of vanishing magnetic field ($B = 0$). For that, one can use the formula for the asymptotic of ψ -function at large values of the argument

$$\psi(z) = \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} + O\left(\frac{1}{z^6}\right) \quad (29)$$

to get

$$\begin{aligned} \kappa_0 &= \frac{N}{4\pi^2 T^2} \int_0^{\infty} \frac{d\omega\omega^2}{\cosh^2 \frac{\omega}{2T}} \\ & \times \left[1 + \frac{\omega^2 - \sigma^2 + \Gamma^2}{2\omega\Gamma} \left(\frac{\pi}{2} - \arctan \frac{\sigma^2 + \Gamma^2 - \omega^2}{2\omega\Gamma} \right) \right]. \end{aligned} \quad (30)$$

This expression describes the behavior of the thermal conductivity as a function of temperature for massive Dirac's particles and is relevant for the supercritical phase of the NJL model, where the mass is generated spontaneously even at zero magnetic field.

The last expression can be evaluated analytically in two regimes, $\Gamma \ll T$:

$$\kappa_0 \simeq \frac{N}{8\pi T^2 \Gamma} \int_{\sigma}^{\infty} d\omega \frac{\omega(\omega^2 - \sigma^2)}{\cosh^2(\omega/2T)} \simeq \frac{N\sigma^2}{\pi\Gamma} e^{-\frac{\sigma}{T}}, \quad T \ll \sigma, \quad (31)$$

and $\Gamma \gg T$:

$$\frac{\kappa_0}{T} \simeq \frac{N}{3} \left[\frac{\Gamma^2}{\sigma^2 + \Gamma^2} + \frac{7\pi^2 T^2 \Gamma^2 (\Gamma^2 + 5\sigma^2)}{15 (\sigma^2 + \Gamma^2)^3} \right]. \quad (32)$$

Accordingly, in the weak coupling phase of the NJL model where the dynamical mass is not generated we obtain:

$$\frac{\kappa_0}{T} = \frac{N}{\pi} \left[\frac{9\zeta(3)}{4} \frac{T}{\Gamma} + \frac{\ln 2}{2} \frac{\Gamma}{T} \right], \quad \Gamma \ll T, \quad (33)$$

and

$$\frac{\kappa_0}{T} = \frac{N}{3} \left[1 + \frac{7\pi^2 T^2}{15 \Gamma^2} \right], \quad \Gamma \gg T. \quad (34)$$

Equations (33, 34) up to an overall factor coincide with the corresponding expressions obtained in reference [26] for the vortex state of nodal quasiparticles in the d -wave superconducting phase of high- T_c cuprates. The overall factor there equals $(v_F^2 + v_{\Delta}^2)/v_F v_{\Delta}$ where v_F, v_{Δ} are respectively the velocities perpendicular and tangential to the Fermi surface. They originate from the quasiparticle excitation spectrum in the vicinity of the gap nodes which takes the form of an anisotropic Dirac cone $E(k) = \sqrt{v_F^2 k_1^2 + v_{\Delta}^2 k_2^2}$. With the overall factor replacing N ($= 2$ in real d -wave superconductor), the first term in equation (34) reproduces the universal (or residual) thermal conductivity at low T in the so-called "dirty" limit, $T \ll \Gamma$, since it is independent of the impurity density, thus it will not depend on the specific characteristics of the scattering processes in the sample. The residual conductivity was recently observed in experiments [27] confirming the existence of gapless quasiparticles in d -wave cuprates at $T < T_c$. Note, however, that in contrast to what was claimed in reference [28], the low-temperature thermal conductivity for massive quasiparticles (Eq. (32)) does not exhibit a universal behavior when $T \rightarrow 0$. This peculiarity of the low-temperature thermal conductivity can be used to find out experimentally the second gap in cuprates.

Expressions (33, 34) were recently used to propose a scenario for the arising of a plateaux at high magnetic fields [26] in Krishana's experiment. In that scenario the width Γ of QPs becomes dependent on the field due to scattering on disordered vortices, thus Γ becomes $\Gamma_0 + \Gamma_B$ where the field induced width Γ_B is calculated to be $\sim \sqrt{B}$. All the information regarding the magnetic field is encoded now in the total width Γ , hence the magnetic field

is not explicitly present. If we start with a weak magnetic field, when $\Gamma_B \ll \Gamma_0 \ll T$, the thermal conductivity follows first the expression (33) (weak field regime) decreasing with the field. At some point, when Γ becomes of order T a crossover takes place to the high field regime (34) with plateaux. Physically, such a scenario is applicable only if there is a small number of vortices with large distances between them and the magnetic field is basically confined in tubes. However, for the field range of interest, $H_{c1} \ll B \ll H_{c2}$, where H_{c1}, H_{c2} are the lower and upper critical magnetic fields of the high T_c superconductor respectively, the vortices are dense enough to overlap strongly giving rise to an effective uniform magnetic field in the whole plane [29], so the above scenario is not perhaps the most appropriate for this field range.

2.3 Non-zero magnetic field

We shall analyze now the thermal conductivity in the presence of a uniform magnetic field in the whole plane. The analysis in all cases will be made at fixed T and Γ , and we do not assume the last one to be dependent on the field.

From the phase transition analysis of the (2+1)-dimensional NJL model (see Appendix A) it follows that at finite temperature there exists a critical value of the magnetic field $B_c(T)$, above which the magnetic catalysis phenomenon occurs generating a dynamical fermion mass even at weak coupling (in what follows we consider only the weak coupling case $g \lesssim g_c$). For magnetic fields less than the critical one ($eB_c(T) \sim 16T^2$) the dynamical mass is zero ($\sigma = 0$).

2.3.1 Narrow width

To study the narrow width limit $\Gamma \rightarrow 0$, we replace the fractions $\Gamma/(\Gamma^2 + x^2)$ in equation (27) by $\pi\delta(x)$. Then, after integrating over ω , what is equivalent to evaluating in the mass shell for the different LL, we obtain

$$\kappa \simeq \frac{N\Gamma}{4\pi T^2} \left\{ \frac{(eB)^2}{(eB)^2 + 4\sigma^2 \Gamma^2} \frac{\sigma^2}{\cosh^2 \frac{\sigma}{2T}} + \sum_{n=1}^{\infty} \frac{(2eB)^2 n (\sigma^2 + 2eBn)}{(eB)^2 + 4(\sigma^2 + 2eBn)\Gamma^2} \frac{1}{\cosh^2 \frac{\sqrt{\sigma^2 + 2eBn}}{2T}} \right\}, \quad \Gamma \rightarrow 0. \quad (35)$$

In (35) we kept Γ^2 in the denominators in order to be able to reproduce a smooth behavior of $\kappa(B)$ in the limit $B \rightarrow 0$. The origin of the first term in (35), whose essential role in the kink-like behavior is discussed below, can be traced back to the leading contribution of the zeroth to first LL transitions. Note that it contributes only when the dynamical mass is present, *i.e.* if $\sigma \neq 0$. That is, because the ratio $\Gamma/((\omega - \sigma)^2 + \Gamma^2)$ appearing in equation (28) becomes $\pi\delta(\omega - \sigma)$ in this limit, it does not contribute to (35) unless $\sigma \neq 0$, due to the presence of the factor ω^2 in the integrand of equation (27). This means that in the

narrow width limit the magnetic catalysis is not only connected to the generation of the mass, but it is responsible also for the enhancement of the transitions between zeroth and first LLs. The fact that the thermal conductivity is proportional in this limit to the scattering rate (width) Γ means that it results from transitions of quasiparticles between cyclotron orbits mediated by scattering of QPs on impurities.

a. Weak field limit, $\sqrt{eB} < 4T$

Taking into account that at weak coupling no dynamical mass is generated for fields below the critical value ($eB < eB_c$), we take $\sigma = 0$ in the calculation that follows. Making use of the Euler-MacLaurin formula

$$\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) \simeq \int_0^{\infty} dx F(x) - \frac{1}{12}F'(0), \quad (36)$$

expression (35) with zero dynamical mass can be recast for subcritical fields $\sqrt{eB} < \sqrt{eB_c} \sim 4T$ in the form

$$\kappa = \frac{2NT^2}{\pi\Gamma} \int_0^{\infty} \frac{dx x^5}{\cosh^2 x} \frac{1}{x^2 + \left(\frac{eB}{4T}\right)^2}. \quad (37)$$

Equation (37) shows monotonic decreasing of κ with increasing magnetic field B (at $B = 0$ it reproduces the leading term in Eq.(33)). Note that the scale $\sqrt{4T\Gamma}$ marks the crossover point where the transition from superweak ($\sqrt{eB} \lesssim \sqrt{4T\Gamma}$) to weak ($\sqrt{4T\Gamma} \lesssim \sqrt{eB} < 4T$) fields takes place.

b. Strong field limit, $\sqrt{eB} \gtrsim 4T$

We shall consider now the strong field regime, $\sqrt{eB} \gtrsim 4T$, where a nonzero dynamical fermion mass is generated in the weakly interacting system (we are interested mainly in the region of coupling constants $\bar{g} \lesssim \bar{g}_c$ where the scaling $\sigma \sim \sqrt{eB}$ is achieved).

Let us start, however, analyzing the case of free massless fermions ($\sigma = 0$). In this case, one can use equation (35), after evaluating it in $\sigma = 0$, to describe the very large field ($\sqrt{eB} \gg 4T$) behavior of κ in the narrow width case, what yields an exponential fall of the conductivity

$$\kappa \simeq \frac{8N\Gamma eB}{\pi T^2} e^{-\frac{\sqrt{2eB}}{T}}. \quad (38)$$

Coming back to the interacting case and after dropping the term depending on Γ in the denominators of equation (35), we obtain

$$\kappa \simeq \frac{N\Gamma}{4\pi T^2} \left\{ \frac{\sigma^2}{\cosh^2 \frac{\sigma}{2T}} + \sum_{n=1}^{\infty} \frac{4n(\sigma^2 + 2eBn)}{\cosh^2 \frac{\sqrt{\sigma^2 + 2eBn}}{2T}} \right\}. \quad (39)$$

In the limit of large fields ($\sqrt{eB} \gg 4T$), the first term in (39) is the leading one. Such a term would produce a sharp plateau, were the fermion mass a constant. Since in the model under consideration the mass is dynamical and it behaves as $\sigma \sim \sqrt{eB}$ at $\sqrt{eB} \gg \sqrt{eB_c} \simeq 4T$, the first term gives rise to an exponential decrease as in the case of free massless fermions (38) for asymptotically large fields.

c. Near the phase transition point, $eB \gtrsim eB_c$

In this case, the thermal conductivity can still be approximated by equation (39). As shown in Appendix A, near the mean field phase transition point $\sigma \approx \frac{1}{2}\sqrt{eB - eB_c}$, so if the field lies in the interval $eB_c < eB \leq 2eB_c$, the dynamical mass $\sigma \lesssim 2T$ and hence the $\cosh^2 \frac{\sigma}{2T}$ appearing in the first term of equation (39) is of order one. In this field region the first term of equation (39) gives a positive contribution to the derivative of κ close to the critical point. That positive contribution leads to a jump in the slope of κ at $eB = eB_c$ therefore showing a kink-like behavior for κ in the narrow-width case.

2.3.2 Finite width

Let us consider now the case where the width Γ is small but finite. We are particularly interested in the behavior of κ near the phase transition point, where $eB \gtrsim eB_c$, and therefore $\sqrt{eB} > 4T$, Γ, σ . From equation (25) one can see that for these fields the contribution of transitions between Landau levels with $n \geq 1$ in the integrand behaves as $1/(eB)^2$, while the transitions between zeroth and first LL decrease as $\sim 1/eB$. However, the LL degeneracy is also proportional to eB , what implies that the transitions between the zeroth and first LL are not suppressed despite the fact that the gap between levels grows with the field. The leading in $1/eB$ behavior is easy to obtain from equation (28). It is given by the expression

$$\kappa = \frac{N\Gamma^2}{4\pi^2 T^2} \int_0^{\infty} \frac{d\omega \omega^2}{\cosh^2 \frac{\omega}{2T}} \left\{ \left(1 + \frac{2\omega(\omega + \sigma)}{eB} \right) \frac{1}{(\omega + \sigma)^2 + \Gamma^2} + \left(1 + \frac{2\omega(\omega - \sigma)}{eB} \right) \frac{1}{(\omega - \sigma)^2 + \Gamma^2} \right\}. \quad (40)$$

As discussed in the previous subsection, near the transition the dynamical mass behaves as $\sigma \approx \frac{1}{2}\sqrt{eB - eB_c}$, so we can expand equation (40) around $\sigma = 0$ to obtain

$$\kappa = \frac{N\Gamma^2}{2\pi^2 T^2} \int_0^{\infty} \frac{d\omega \omega^2}{\cosh^2 \frac{\omega}{2T}} \left\{ \frac{1 + 2\omega^2/eB}{\omega^2 + \Gamma^2} + \frac{\sigma^2}{(\omega^2 + \Gamma^2)^2} \left[\frac{3\omega^2 - \Gamma^2}{\omega^2 + \Gamma^2} + \frac{2\omega^2}{eB} \left(\frac{\omega^2 - 3\Gamma^2}{\omega^2 + \Gamma^2} \right) \right] \right\}. \quad (41)$$

Clearly, for $\Gamma < \sqrt{3}(2T)$, the term proportional to σ^2 gives a positive contribution to the derivative of κ with respect to B near the transition point, so there is a jump in the slope of κ at the critical point: a kink-like effect. Notice that if σ were zero or constant, the derivative of the thermal conductivity would satisfy $\frac{d\kappa}{deB} \simeq \frac{-C}{(eB)^2}$, with C positive, so no kink-like effect would be present. On the other hand, since σ is dynamical, near the transition point $\frac{d\kappa}{deB} \simeq \alpha - \beta/eB + O((eB)^{-2})$, with α and β positive, hence one can see that the dynamical mass not only allows for a jump in the slope, but it flattens the profile after

the critical field, allowing at least for certain region of the parameter space a behavior of almost zero slope: a plateau-like profile. This kink and plateau-like behavior is corroborated by numerical calculations in the next section.

We highlight that the dynamical mass is needed to obtain the kink-like effect in both narrow and finite width cases. Hence, in our model the kink of the thermal conductivity is directly linked to the magnetic catalysis phenomenon. Moreover, any model with the same critical behavior for σ would lead to a similar effect. This means that our results are indeed model independent, since any relativistic theory of interacting fermions that belongs to the universality class determined by the critical behavior here considered would yield a similar kink-like feature.

One should note that the mean field behavior of the dynamical mass $\sigma \sim \sqrt{eB - eB_c}$ may change if higher order corrections (fluctuations) are taken into account in the gap equation. The fluctuations could either change the phase transition to a first order one, with a discontinuity in σ at the phase transition point (this was a suggestion made by Laughlin in [36]), or to a non-mean-field order phase transition, with the scaling law $\sigma \sim (eB - eB_c)^\nu$ where $\nu > 1/2$. While in the former case a discontinuity will appear in the thermal conductivity, in the latter case the conductivity will be a smooth function of the magnetic field, and a singularity will move to its higher derivatives.

2.3.3 Low temperature limit

Finally, we give an expression for the thermal conductivity when the temperature is much less than both Γ and eB . At low T the function $\cosh^{-2}(\omega/2T)$ in equation (27) is very sharply peaked at $\omega = 0$, thus, expanding the rest of the integrand over ω and performing the integration we get

$$\begin{aligned} \kappa = \frac{NT}{3} & \left\{ \frac{\Gamma^2}{\sigma^2 + \Gamma^2} + \frac{7\pi^2 T^2 \Gamma^2}{5} \right. \\ & \times \left[\frac{3\sigma^2 - \Gamma^2}{(\sigma^2 + \Gamma^2)^3} + \frac{2}{eB} \frac{\sigma^2 - \Gamma^2}{(\sigma^2 + \Gamma^2)^2} \right. \\ & \left. \left. + \frac{2}{(eB)^2} \frac{\sigma^2 - \Gamma^2}{\sigma^2 + \Gamma^2} - \frac{\sigma^2 - \Gamma^2}{(eB)^3} \psi' \left(\frac{\sigma^2 + \Gamma^2}{2eB} \right) \right] \right\}. \quad (42) \end{aligned}$$

It is easy to see, using the asymptotic of the ψ -function at large values of its argument, that when $B \rightarrow 0$ the expression (42) goes to equation (32) in spite of the fact that we approach $B = 0$ from the side $B > T$. For large fields we get

$$\begin{aligned} \kappa = \frac{NT}{3} & \left\{ \frac{\Gamma^2}{\sigma^2 + \Gamma^2} + \frac{7\pi^2 T^2 \Gamma^2}{5} \right. \\ & \left. \times \left[\frac{3\sigma^2 - \Gamma^2}{(\sigma^2 + \Gamma^2)^3} - \frac{2}{eB} \frac{\sigma^2 - \Gamma^2}{(\sigma^2 + \Gamma^2)^2} \right] \right\}. \quad (43) \end{aligned}$$

Note that at large eB the thermal conductivity would approach some constant value in the case of constant mass,

if $\sigma > \Gamma$, $\kappa(B)$ approaches that asymptotical value from below. This resembles the low T behavior of the thermal conductivity in d -wave cuprates [30]. On the other hand, if the mass σ is a dynamical one with asymptotic behavior $\sigma \sim \sqrt{eB}$ as in our four-fermion model, then $\kappa(B)$ goes to zero as $1/eB$ at large fields. This is different from the thermal conductivity behavior for massless particles which, neglecting T^3 corrections, tends to the universal constant $\kappa = NT/3$ for $\sigma = 0$ at large B (see, Eq. (43)).

3 Thermal conductivity profile: numerical calculations

In this section we do a numerical study of the profile of the thermal conductivity *versus* the applied magnetic field, taking into account the generation of the dynamical mass at a critical field that depends on the temperature. The field-dependence of the finite-temperature dynamical mass is obtained from the solution of the gap equation (A.4) derived in Appendix A.

To numerically investigate the behavior of the thermal conductivity (27) within a parameter range that can be of interest for condensed matter applications, we need to restore all the model parameters, like $\hbar, c, k_B, v_F, v_\Delta$. Following reference [11] we write the Lagrangian density as

$$\begin{aligned} \mathcal{L} = \frac{1}{2} & \left[\bar{\psi}_i, \left(i\gamma^0 \hbar \frac{\partial}{\partial t} + v\gamma^j \left(i\hbar \frac{\partial}{\partial x^j} - \frac{e}{c} A_j \right) \right) \psi_i \right] \\ & + \frac{gv}{2N} (\bar{\psi}_i \psi_i)^2, \quad (44) \end{aligned}$$

where v_F and v_Δ entering in $v = \sqrt{v_F v_\Delta}$ were defined in the previous section, and the external potential is given by equation (2). As known, this Lagrangian is equivalent to

$$\begin{aligned} \mathcal{L} = \frac{1}{2} & \left[\bar{\psi}_i, \left(i\gamma^0 \hbar \frac{\partial}{\partial t} + v\gamma^j \left(i\hbar \frac{\partial}{\partial x^j} - \frac{e}{c} A_j \right) \right) \psi_i \right] \\ & - \sigma v \bar{\psi}_i \psi_i - \frac{N\sigma^2 v}{2g}, \quad (45) \end{aligned}$$

since the Euler-Lagrange equation for the auxiliary scalar field σ obeys the constraint $\sigma = -(g/N)\bar{\psi}_i \psi_i$ so that the Lagrangian density (45) reproduces equation (44) upon application of this constraint. The effective action for the composite field σ can be obtained by integrating over fermions in the path integral. From the minimum condition of the effective potential $V(\sigma)$ one finds that, at fixed T , there is a critical value of the magnetic field $\sqrt{eB_c}/T \simeq 4.1476$ such that for subcritical fields $eB \leq eB_c$ the gap is zero, while for $eB > eB_c$ it is given by the non-trivial solution of the gap equation (A.4) (see Appendix A).

Notice that σv has dimension of energy and plays the role of mc^2 in the Dirac Lagrangian density. To generate a plot of κ/κ_0 (κ_0 is the thermal conductivity at zero field) *versus* the magnetic field, we need to substitute $T \rightarrow k_B T, \Gamma \rightarrow \hbar\Gamma, eB \rightarrow (\hbar v^2/c)eB$, where B is measured in

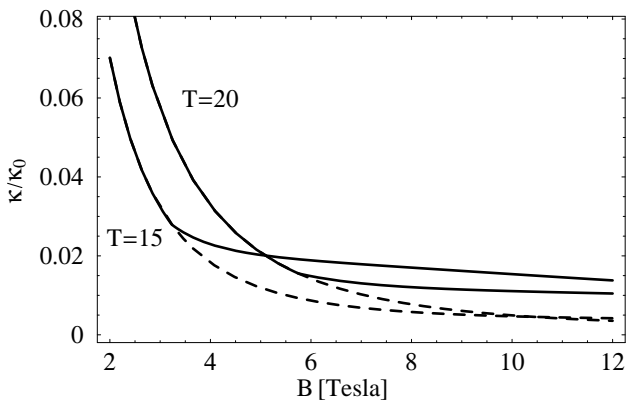


Fig. 1. The magnetic field dependence of κ at $T = 20$ K and $T = 15$ K in the narrow width case ($\Gamma = 5$ K). The solid lines represent κ/κ_0 when a QP gap σ is MC-induced at $B \geq B_c(T)$ ($B_c(20) = 5.75$ T, $B_c(15) = 3.23$ T). The dashed lines represent the behavior of κ/κ_0 when σ remains zero at $B \geq B_c(T)$.

Gauss. It is convenient to measure all energetic quantities in degrees of K, what leads to the replacement $eB \rightarrow 0.8 \times 10^{10}(v/c)^2 \text{ }^\circ\text{K}^2 \times B(\text{tesla})$, where the magnetic field is measured now in tesla's. Thus, using the approximated value of the characteristic velocity $v_D \simeq 10^7$ cm/s [11–18], $eB \rightarrow 2.92 \times 10^2 \times \text{ }^\circ\text{K}^2 \times B(\text{tesla})$, and we obtain the critical curve $B = 0.014T^2$, which fits the experimental curve of reference [14].

Let us numerically find the profile of κ versus the magnetic field at a fixed T . In Figure 1 $\kappa(B)$ has been plotted for two different temperatures taking into account the generation of the dynamical mass for $eB > eB_c$. Here we corroborate what we had already argued in Section 2 based on the analytic result found for κ : due to the appearance of σ at a critical field that depends on the temperature, κ exhibits a kink behavior in its profile with the magnetic field. Moreover, for $B > B_c$ the kink is followed by a region where κ is only weakly dependent on the field (plateau-like region). With decreasing temperature, the position of the kink moves to the left in accordance with the critical line $B_c = 0.014T^2$.

The numerical calculations revealed the sensitivity of the kink-plateau feature of the thermal conductivity to the relation between Γ and T . Only when Γ was not much smaller than T the thermal conductivity showed a kink-plateau profile (in Fig. 1 the curves shown correspond to the ratios $\Gamma/T = 0.25$ and 0.33).

4 Concluding remarks

In the present paper we studied the thermal conductivity of relativistic fermions in a (2+1)-dimensional four-fermion interaction model as a function of the applied magnetic field, the temperature and the particle width. We have shown that, for certain relations among these parameters, the profile of the thermal conductivity versus

the applied field exhibits a kink-like behavior at $B \simeq B_c$, where B_c is the critical field for the generation of a fermion dynamical mass σ , followed by a plateau-like region at $B \geq B_c$. We point out that the kink effect is the consequence of two main features: the generation of a fermion gap in the presence of the magnetic field (MC phenomenon), and the enhancement of the zeroth-to-first LLs transitions.

A main outcome of our investigation is that the relevant properties of the thermal conductivity of the (2+1)-dimensional relativistic QP system around the critical point are model independent. Indeed, the essential ingredient of the effective model required to produce the kink-like effect in the thermal conductivity is the critical behavior of the dynamical mass induced by MC near the phase transition, so not much depends on the concrete form of the effective Lagrangian. This fact makes our result relevant beyond the particular model under consideration, linking it to the universality class of theories with such a critical behavior. Such an universal character opens a window for possible applications.

From a quantum field theory viewpoint, condensed matter systems whose Fermi surfaces are only characterized by nodal points are especially interesting for us, since at low energies they can be described by relativistic quantum field theory models of massless fermions [32]. Along that direction, a feasible possibility for the application of our results is the heat transport properties of graphite in the presence of a magnetic field. Let us recall that HOPG materials [19] have layered structure with two isolated points in the Brillouin zone where the dispersion is linear. Their electronic states can be thus described in terms of relativistic charged particles [31]. This graphite could exhibit the phenomenon of MC as suggested in reference [18] to explain the semimetal-insulator phase transition observed in HOPG in the presence of a magnetic field perpendicular to the layers. As the quasiparticles in the graphite are subjected to Landau quantization under a perpendicularly applied magnetic field, our results should have full strength there and we anticipate that the thermal conductivity of these systems will show a behavior similar to the one reported in the present work (for computation of the electric conductivity in graphite along the lines followed in the present work, see recent paper [33]).

On the other hand, the characteristic feature of d -wave superconductors is also the existence of nodal points (four in this case) where the order parameter vanishes, thus the Fermi surface consists of four isolated points with excitations around them being well-defined gapless quasiparticles (QP). The kinetic part of the QPs effective Lagrangian is nothing but the Dirac Lagrangian for two species of massless four-component spinors [34]. At low temperatures such QPs give the main contribution to thermodynamic and transport properties. There is now considerable experimental evidence for the existence of well-defined QPs in the superconducting state of cuprates (see Ref. [35] and references therein).

As was mentioned in the Introduction, the MC has been suggested [8–11] to be behind the odd behavior of the

thermal conductivity of high- T_c superconducting cuprates in a magnetic field observed in the experiment [14, 15, 17], although alternative solutions, not based on MC, has also been proposed [26, 36]. According to the experiment done by Krishana *et al.* [14], and later reproduced by other authors [15, 17], at temperatures significantly lower than T_c of superconductivity, the thermal conductivity $\kappa(B)$, as a function of a magnetic field perpendicularly applied to the cuprate planes of the samples, displays a sharp break in its slope (kink-like behavior) at a transition field B_κ , followed by a plateau region in which it ceases to change with increasing field up to the highest attainable fields ~ 14 T. The critical temperature for the appearance of the kink-like behavior scales with the magnetic field as $T_\kappa \sim \sqrt{B}$ similar to the scaling of the critical temperature with the field found in NJL models [2]. It is worthy to mention here that the reliability of the thermal conductivity experiments of Krishana *et al.* [14] has been a matter of debate in the last years [16], although it seems that finally the contradictory results have been clarified and understood [17].

We emphasize that none of the previously mentioned [8–11, 26, 36] attempted explanations of Krishana's experiment were able to obtain on a theoretical basis the kink-like behavior observed in the experiment [14]. In the case of our results, although it is striking that, as discussed in our previous paper [12] and shown in Figure 1, our numerical curves exhibit a profile of the thermal conductivity with the applied magnetic field with similar qualitative characteristics³ to those observed in the experiment [14], we cannot claim that our results are valid in the vortex state of the superconductor since our calculations are based on the Landau level quantization, whose applicability to the d-wave superconductor in the presence of vortices has been recently subjected to intense criticism [37–39].

Nevertheless, the mechanism of generating a kink-like effect in the thermal conductivity via the MC phenomenon, as studied in the previous sections, should be relevant for condensed matter systems with Dirac-like charged QP on which Landau level quantization is feasible. In this direction, we would like to point out that there is still some chance for the realization of MC in cuprate systems. As it has been recently argued in references [40, 41], the physical picture underlying the description of superconducting cuprates may involve the interplay of two different phases with their corresponding order parameters: one $d_{x^2-y^2}$ superconducting (DSC) and one of density wave order (DDW). The DDW state, unlike the DSC state, does not break gauge invariance, so the quasiparticles of the DDW state can form Landau levels under an applied magnetic field [41]. Although the above scenario is still speculative and more experimental confir-

mation is required, one can venture that given that the charged QP excitations around the nodes of the DDW order parameter have Landau levels, they can be subjected to a MC phase transition at some critical magnetic field, and hence our theoretical results could be relevant in that case for the description of the heat transport properties of the DDW-state in the magnetic field.

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Appendix A: Dynamical mass scaling near the phase transition point

The effective potential of the (2+1)-dimensional NJL model in a constant external magnetic field at finite temperature was computed in [2]. The integration over fermion fields in the functional integral with the Lagrangian (4) can be performed to give the effective potential in the form

$$\begin{aligned} V(\sigma) &= \frac{N\sigma^2}{2g} + \frac{NeB}{4\pi^{3/2}} \\ &\times \int_0^\infty \frac{dt}{t^{3/2}} e^{-t\sigma^2} \coth eBt \Theta_4 \left(0 \middle| \frac{i}{4\pi T^2 t} \right) \\ &= V_{0,B}(\sigma) + V_{T,B}(\sigma), \end{aligned} \quad (\text{A.1})$$

where the temperature independent part of the potential is given by

$$\begin{aligned} V_{0,B}(\sigma) &= \frac{N\sigma^2}{2g} + \frac{NeB}{4\pi^{3/2}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-t\sigma^2} \coth eBt \\ &= \frac{N}{\pi} \left[\frac{1}{2} M_0 \sigma^2 - \sqrt{2} (eB)^{3/2} \zeta \left(-\frac{1}{2}, \frac{\sigma^2}{2eB} + 1 \right) - \frac{\sigma eB}{2} \right], \end{aligned} \quad (\text{A.2})$$

(the mass scale parameter $M_0 = \pi/g - \Lambda/\sqrt{\pi}$ and Λ is the ultraviolet cutoff which is taken much bigger than all other parameters in the model) and the part depending on temperature is

$$\begin{aligned} V_{T,B}(\sigma) &= \\ &= \frac{NeB}{4\pi^{3/2}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-t\sigma^2} \coth eBt \left[\Theta_4 \left(0 \middle| \frac{i}{4\pi T^2 t} \right) - 1 \right]. \end{aligned} \quad (\text{A.3})$$

Here $\theta_4(v|\tau)$ is the Jacobi's elliptic function.

³ In addition to the kink-plateau behavior and the scaling of the critical field with the temperature, our curves have another similarity with the experimental behavior reported in [14], namely, with decreasing T the crossing of the curves occurs in such a way that the lower T curve reaches the higher value at large fields.

The gap equation $dV(\sigma)/d\sigma = 0$ follows from (A.1)

$$\sigma \left[-\frac{M_0}{\sqrt{eB}} + \frac{1}{\sqrt{2}} \zeta \left(\frac{1}{2}, \frac{\sigma^2}{2eB} + 1 \right) + \frac{\sqrt{eB}}{2\sigma} \tanh \frac{\sigma}{2T} + \int_0^\infty \frac{dt}{\sqrt{\pi t}} \frac{e^{-\frac{\sigma^2 t}{eB}}}{e^{2t} - 1} \left[\theta_4 \left(0 \middle| \frac{ieB}{4\pi T^2 t} \right) - 1 \right] \right]. \quad (\text{A.4})$$

The solution of this equation defines the fermion dynamical mass. At $T = 0, B = 0$ the gap equation admits a nontrivial solution only if the coupling g is supercritical, $g > g_c = \pi^{3/2}/\Lambda$ ($M_0 < 0$). The gap equation at $T = 0, B \neq 0$ was also studied in the literature (see, for example, Refs. [1,2]). It was shown that it always has a nontrivial solution at all $g > 0$ no matter how small the magnetic field B might be. In the weak coupling phase the field-induced dynamical mass at zero temperature behaves as $\sigma_0 = eB/2M_0$ in weak fields ($\sqrt{eB} \ll M_0$), while at high fields ($\sqrt{eB} \gg M_0$) it is $\sigma_0 \simeq 0.446\sqrt{eB}$. At finite temperature the critical line in the $B - T$ plane separating the massless and massive phases was calculated numerically in [2] (see also [11]). We shall derive here an analytical solution of the gap equation near such a critical line.

We start, actually, with the derivation of the Landau-Ginzburg-like potential by expanding $V(\sigma)$ in powers of σ , since near the phase transition point σ is small, and write

$$V(\sigma) = V(0) - \frac{1}{2}M(T, eB)\sigma^2 + \frac{1}{4}\lambda(T, eB)\sigma^4.$$

As it is accustomed for second-order phase transitions, the equality of the coefficient $M(T, eB)$ to zero defines the phase transition curve. The region of parameters T, B where $M(T, eB) > 0$ corresponds to the spontaneously broken phase with fermions acquiring the mass, whereas the region $M(T, eB) < 0$ corresponds to the massless phase. Our goal is to obtain the coefficients $M(T, eB)$, and $\lambda(T, eB)$. When it is done the solution of the gap equation is given by

$$\sigma^2 = \frac{M(T, eB)}{\lambda(T, eB)}. \quad (\text{A.5})$$

From (A.5) we obtain the behavior of the dynamical mass near the phase transition point ($M(T_c, eB_c) = 0$). In particular, at fixed temperature and $B \rightarrow eB_c$ we find

$$\begin{aligned} \sigma^2(T = T_c, eB) &= \frac{M(T_c, eB)}{\lambda(T_c, eB)} \\ &\simeq \frac{d}{deB} \left(\frac{M(T_c, eB)}{\lambda(T_c, eB)} \right)_{B=eB_c} (eB - eB_c) \\ &\simeq \frac{1}{\lambda(T_c, eB_c)} \frac{d}{deB} M(T_c, eB)|_{eB=eB_c} (eB - eB_c). \end{aligned} \quad (\text{A.6})$$

The potential $V_{0,B}(\sigma)$ is easily expanded in powers of σ^2 . Let us turn to $V_{T,B}(\sigma)$ which we write as the sum of two

terms

$$\begin{aligned} V_{T,B}(\sigma) &= \frac{NeB}{4\pi^{3/2}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-t\sigma^2} \left[\Theta_4 \left(0 \middle| \frac{i}{4\pi T^2 t} \right) - 1 \right] \\ &+ \frac{NeB}{4\pi^{3/2}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-t\sigma^2} (\coth eBt - 1) \left[\Theta_4 \left(0 \middle| \frac{i}{4\pi T^2 t} \right) - 1 \right]. \end{aligned} \quad (\text{A.7})$$

The second term in the last expression can be expanded in a series in σ^2 since the two brackets (with cotangent and θ -function) regularize the behavior of the integrand at infinity and zero, respectively. Thus we need to calculate the first term

$$\begin{aligned} \int_0^\infty \frac{dt}{t^{3/2}} e^{-t\sigma^2} \left[\Theta_4 \left(0 \middle| \frac{i}{4\pi T^2 t} \right) - 1 \right] &= \\ &2 \sum_{n=1}^\infty (-1)^n \int_0^\infty \frac{dt}{t^{3/2}} e^{-t\sigma^2 - \frac{\pi^2}{4T^2 t}} \\ &= 4\sqrt{2T}\sigma \sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n}} K_{1/2} \left(\frac{n\sigma}{T} \right) = -4\sqrt{\pi}T \log(1 + e^{-\frac{\sigma}{T}}), \end{aligned} \quad (\text{A.8})$$

where $K_\nu(z)$ is a modified Bessel function ($K_{1/2}(z) = (\pi/2z)^{1/2}e^{-z}$).

Finally, we obtain the following expressions for the coefficients

$$\begin{aligned} M(T, B) &= \frac{N\sqrt{eB}}{\pi} \left\{ -\frac{M_0}{\sqrt{eB}} + \frac{\zeta(1/2)}{\sqrt{2}} + \frac{\sqrt{eB}}{4T} + \frac{\sqrt{eB}}{2\sqrt{\pi}} \int_0^\infty \frac{dt}{t^{1/2}} (\coth eBt - 1) \right. \\ &\times \left. \left[\Theta_4 \left(0 \middle| \frac{i}{4\pi T^2 t} \right) - 1 \right] \right\}, \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} \lambda(T, B) &= \frac{N}{\pi\sqrt{eB}} \left\{ \frac{\zeta(3/2)}{4\sqrt{2}} + \frac{1}{48} \left(\frac{\sqrt{eB}}{T} \right)^3 + \frac{(eB)^{3/2}}{2\pi} \right. \\ &\times \int_0^\infty dt t^{1/2} (\coth eBt - 1) \\ &\times \left. \left[\Theta_4 \left(0 \middle| \frac{i}{4\pi T^2 t} \right) - 1 \right] \right\}. \end{aligned} \quad (\text{A.10})$$

The critical curve $M(T, B) = 0$ can be analyzed analytically at $T \ll \sqrt{eB}$ where the integral in equation (A.9) is exponentially small and for T_c we get the equation

$$\frac{\sqrt{eB}}{4T_c} = \frac{M_0}{\sqrt{eB}} - \frac{\zeta(1/2)}{\sqrt{2}}. \quad (\text{A.11})$$

For the solution to exist, it must satisfy that $\sqrt{eB} \ll M_0$, what gives the critical temperature

$$T_c \simeq \frac{eB}{4M_0} = \frac{1}{2}\sigma_0, \quad (\text{A.12})$$

where $\sigma_0 \equiv \sigma(T = 0)$ is the dynamical mass at zero temperature.

One can convince oneself that there is no solution of the equation $M(T, B) = 0$ when $T_c \gg \sqrt{eB}$. Indeed, for that let us write the integral in (A.9) as

$$\begin{aligned} I &= \int_0^\infty \frac{dt}{t^{1/2}} (\coth eBt - 1) \left[\Theta_4\left(0 \middle| \frac{i}{4\pi T^2 t}\right) - 1 \right] \\ &= \frac{1}{\sqrt{eB}} \int_0^\infty \frac{dt}{t^{1/2}} (\coth t - 1) \left[\Theta_4\left(0 \middle| \frac{ieB}{4\pi T^2 t}\right) - 1 \right]. \end{aligned} \quad (\text{A.13})$$

We further divide the integrand into three pieces

$$\begin{aligned} I &= \frac{1}{\sqrt{eB}} \left\{ \int_0^\infty \frac{dt}{t^{3/2}} \left[\Theta_4\left(0 \middle| \frac{ieB}{4\pi T^2 t}\right) - 1 \right] \right. \\ &\quad - \int_0^\infty \frac{dt}{t^{1/2}} \left(\coth t - \frac{1}{t} - 1 \right) \\ &\quad \left. + \int_0^\infty \frac{dt}{t^{1/2}} \left(\coth t - \frac{1}{t} - 1 \right) \Theta_4\left(0 \middle| \frac{ieB}{4\pi T^2 t}\right) \right\}. \end{aligned} \quad (\text{A.14})$$

The first and second integrals in the last expression can be evaluated exactly (after changing the variable $t \rightarrow x^2$ in the first integral) with the help of the formulas [42]

$$\int_0^\infty \frac{dt}{t^{3/2}} \left[\Theta_4\left(0 \middle| \frac{ieB}{4\pi T^2 t}\right) - 1 \right] = -4\sqrt{\pi} \log 2 \frac{T}{\sqrt{eB}}; \quad (\text{A.15})$$

$$\int_0^\infty \frac{dt}{t^{1/2}} \left(\coth t - \frac{1}{t} - 1 \right) = \sqrt{2\pi} \zeta\left(\frac{1}{2}\right). \quad (\text{A.16})$$

The integral with Θ_4 -function is calculated using the Jacobi imaginary transformation to Θ_2 -function and keeping in it only the first term in the series when $eB \rightarrow 0$:

$$\begin{aligned} \Theta_4\left(0 \middle| \frac{ieB}{4\pi T^2 t}\right) &= \sqrt{\frac{4\pi T^2 t}{eB}} \Theta_2\left(0 \middle| \frac{4i\pi T^2 t}{eB}\right) \\ &\simeq 4\sqrt{\frac{\pi T^2 t}{eB}} e^{-\frac{\pi^2 T^2 t}{eB}}. \end{aligned}$$

This reduces the third integral to

$$\begin{aligned} \int_0^\infty \frac{dt}{t^{1/2}} \left(\coth t - \frac{1}{t} - 1 \right) \Theta_4\left(0 \middle| \frac{ieB}{4\pi T^2 t}\right) &\simeq \\ \frac{4T\sqrt{\pi}}{\sqrt{eB}} \int_0^\infty dt \left(\coth t - \frac{1}{t} - 1 \right) e^{-\frac{\pi^2 T^2 t}{eB}} &\simeq \\ -\frac{4\sqrt{\pi eB}}{\pi^2 T}, \quad B \rightarrow 0 \end{aligned} \quad (\text{A.17})$$

(when proceeding to the last equality we changed the variable t to eBt and then expanded over eB).

Thus, combining all formulas we get the following expression for the M -function:

$$M(T, B) \simeq \frac{N\sqrt{eB}}{\pi} \left\{ -\frac{M_0}{\sqrt{eB}} + \frac{\sqrt{eB}}{4T} - 2 \log 2 \frac{T}{\sqrt{eB}} - \frac{2}{\pi^2} \frac{\sqrt{eB}}{T} \right\}.$$

As seen, there is no solution as $eB \rightarrow 0$ for $M_0 > 0$. (In case $eB = 0$ and $M_0 < 0$ we get the standard expression for the critical temperature $T_c = |M_0|/2 \log 2$ [20]). Hence, we arrive at the conclusion that the only remaining possibility is that the root of the equation $M(T, B) = 0$ is of the order of $T_c \simeq \sqrt{eB}$. However, in this case we cannot expand the integral in equation (A.9) and should turn to a numerical calculation. The root of the function $M(T, B)$ when the parameter $M_0 \simeq 0$ is found to be $\sqrt{eB}/T \simeq 4.1476$ what defines the critical line. For the critical temperature this gives $T_c \simeq 0.54\sigma_0$ in agreement with the result of reference [2]. We calculated numerically the coefficient before $eB - eB_c$ in equation (A.6) which is found to be 0.2738, thus the scaling of the dynamical mass near the critical line is given by the formula

$$\sigma \simeq 0.523 \sqrt{eB - eB_c}. \quad (\text{A.18})$$

Appendix B: Kubo formula

For the sake of completeness we derive here the expression for the thermal conductivity in the two-dimensional case used in Section 2. We start from the Kubo's formula for the thermal conductivity tensor [43]

$$\kappa_{ij}(\omega) = \frac{1}{VT} \int_0^\infty dt \int_0^\beta d\lambda \text{Tr} \{ \rho U_j(0) U_i(t + i\lambda) \} e^{-i\omega t}, \quad (\text{B.1})$$

where V is the space volume, T the absolute temperature, ρ is the density matrix, and U_i are the heat current operators with

$$U_i(t) = e^{iHt} U_i e^{-iHt}. \quad (\text{B.2})$$

Integrating over t by parts in equation (B.1), and taking into account that the currents go to zero at $t \rightarrow \infty$, we obtain

$$\begin{aligned} \kappa_{ij}(\omega) &= \frac{1}{VT} \int_0^\infty dt \frac{e^{-i\omega t} - 1}{i\omega} \int_0^\beta d\lambda \frac{\partial}{\partial t} \text{Tr} \{ \rho U_j(0) U_i(t + i\lambda) \} \\ &= \frac{1}{VT} \int_0^\infty dt \frac{e^{-i\omega t} - 1}{\omega} \text{Tr} \{ \rho U_j(0) [U_i(t) - U_i(t + i\beta)] \}, \end{aligned} \quad (\text{B.3})$$

where we used also the fact that the quantity under Tr is a function only of $t + i\lambda$.

Now, taking into account that

$$\begin{aligned} \text{Tr} \{ \rho U_j(0) U_i(t + i\beta) \} &= \\ \text{Tr} \left\{ \frac{1}{Z} e^{-\beta H} U_j(0) e^{iHt - \beta H} U_i(0) e^{-iHt + \beta H} \right\} &= \\ = \text{Tr} \{ \rho U_i(t) U_j(0) \}, \end{aligned} \quad (\text{B.4})$$

we obtain from (B.3) and (B.4) the known expression [44]

$$\kappa_{ij}(\omega) = -\frac{1}{VT} \int_0^\infty dt \frac{e^{-i\omega t} - 1}{\omega} \text{Tr}\{\rho[U_i(t), U_j(0)]\}. \quad (\text{B.5})$$

Using that

$$\text{Tr}\{\rho U_i(t) U_j(0)\}^\dagger = \text{Tr}\{U_j^\dagger(0) U_i^\dagger(t) \rho^\dagger\} = \text{Tr}\{\rho U_j(0) U_i(t)\} \quad (\text{B.6})$$

we have

$$\begin{aligned} \text{Tr}\{\rho[U_i(t), U_j(0)]\} &= \text{Tr}\{\rho U_i(t) U_j(0)\} - \text{Tr}\{\rho U_i(t) U_j(0)\}^\dagger \\ &= 2i \text{Im} \text{Tr}\{\rho U_i(t) U_j(0)\}. \end{aligned} \quad (\text{B.7})$$

Then, we can write (B.5) as

$$\kappa_{ij}(\omega) = -\frac{2i}{VT} \int_0^\infty dt \frac{e^{-i\omega t} - 1}{\omega} \text{Im} \text{Tr}\{\rho U_i(t) U_j(0)\}. \quad (\text{B.8})$$

The thermal conductivity for an isotropic system is given by $\kappa = \kappa_{ii}(0)/d$ where the summation over repeated indices is understood (d is the number of space dimensions, in our case $d = 2$). Equation (B.8) then takes the form

$$\begin{aligned} \kappa &= -\frac{i}{VT} \int_0^\infty dt \lim_{\omega \rightarrow 0} \left(\frac{e^{-i\omega t} - 1}{\omega} \right) \text{Im} \text{Tr}\{\rho U_i(t) U_i(0)\} \\ &= -\frac{1}{VT} \text{Im} \int_0^\infty dt t \text{Tr}\{\rho U_i(t) U_i(0)\}, \end{aligned} \quad (\text{B.9})$$

which is equivalent to equation (6), or,

$$\kappa = \frac{i}{2VT} \int_0^\infty dt t \text{Tr}\{\rho[U_i(t), U_j(0)]\}. \quad (\text{B.10})$$

In the representation of the Hamiltonian eigenfunctions, $e^{-iHt} |n\rangle = e^{-iE_n t} |n\rangle$ we can write

$$\begin{aligned} \text{Tr}\{\rho U_i(t) U_i(0)\} &= \\ &= \sum_{n,m} \frac{1}{Z} \left\{ e^{iHt + \beta H} U_i e^{-iHt} |n\rangle \langle n | U_i | m\rangle \langle m | \right\} \\ &= \sum_{n,m} \frac{1}{Z} e^{-\beta E_n + i(E_n - E_m)t} |\langle n | U_i | m\rangle|^2, \end{aligned} \quad (\text{B.11})$$

where the hermiticity of the heat current operators was used. Similarly

$$\begin{aligned} \text{Tr}\{\rho U_i(0) U_i(t)\} &= \\ &= \sum_{n,m} \frac{1}{Z} e^{-\beta E_n + i(E_m - E_n)t} |\langle n | U_i | m\rangle|^2. \end{aligned} \quad (\text{B.12})$$

Now using equations (B.11) and (B.12) and the symmetry of the matrix elements under the interchange $n \leftrightarrow m$ we can write the correlator function in the form

$$\begin{aligned} \mathcal{G}(t) &= \text{Tr}\{\rho[U_i(t), U_i(0)]\} = \frac{1}{Z} \sum_{n,m} e^{-\beta E_n + i(E_n - E_m)t} \\ &\quad \times \left(1 - e^{-\beta(E_m - E_n)} \right) |\langle n | U_i | m\rangle|^2. \end{aligned} \quad (\text{B.13})$$

The retarded Fourier transform of $\mathcal{G}(t)$ is given by

$$\begin{aligned} \mathcal{G}(\Omega) &= \frac{1}{2\pi} \lim_{\eta \rightarrow 0} \int_{-\infty}^\infty \theta(t) \mathcal{G}(t) e^{i\Omega t - \eta|t|} dt \\ &= \frac{1}{2\pi} \frac{1}{Z} \sum_{n,m} e^{-\beta E_n} \left(1 - e^{-\beta(E_m - E_n)} \right) |\langle n | U_i | m\rangle|^2 \\ &\quad \times \lim_{\eta \rightarrow 0} \int_0^\infty dt e^{i(E_n - E_m + \Omega)t - \eta t} \\ &= \frac{1}{2\pi i} \frac{1}{Z} \lim_{\eta \rightarrow 0} \sum_{n,m} e^{-\beta E_n} \left(1 - e^{-\beta(E_m - E_n)} \right) \\ &\quad \times |\langle n | U_i | m\rangle|^2 \frac{1}{E_m - E_n - \Omega - i\eta}. \end{aligned} \quad (\text{B.14})$$

To obtain the spectral representation of $\mathcal{G}(\Omega)$ we define

$$\begin{aligned} \Theta(\Omega) &= \lim_{\eta \rightarrow 0} [\mathcal{G}(\Omega + i\eta) - \mathcal{G}(\Omega - i\eta)] \\ &= \sum_{n,m} \frac{1}{Z} e^{-\beta E_n} \left(1 - e^{-\beta(E_m - E_n)} \right) |\langle n | U_i | m\rangle|^2 \\ &\quad \times \delta(E_m - E_n - \Omega). \end{aligned} \quad (\text{B.15})$$

From (B.14) and (B.15) we can write

$$\mathcal{G}(\Omega) = \lim_{\eta \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^\infty \Theta(\nu) \frac{d\nu}{\nu - \Omega - i\eta}. \quad (\text{B.16})$$

We can use this representation in order to express the thermal conductivity in terms of the spectral density $\Theta(\Omega)$, which is an important step for the calculations in the Green's function formalism. Using the inverse Fourier transform, we can write

$$\begin{aligned} \mathcal{G}(t) &= \int_{-\infty}^\infty \mathcal{G}(\Omega) e^{-i\Omega t} d\Omega \\ &= \lim_{\eta \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^\infty e^{-i\Omega t} d\Omega \int_{-\infty}^\infty \Theta(\nu) \frac{d\nu}{\nu - \Omega - i\eta}, \quad t > 0, \end{aligned} \quad (\text{B.17})$$

and considering the integration formula

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - i\eta} d\omega = \begin{cases} 2\pi i e^{-\eta t}, & t > 0 \\ 0, & t < 0 \end{cases}, \quad (\text{B.18})$$

the function $\mathcal{G}(t)$ can be transformed to

$$\begin{aligned} \mathcal{G}(t) &= \lim_{\eta \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Theta(\nu) d\nu \int_{-\infty}^{\infty} \frac{e^{-i\Omega t}}{\nu - \Omega - i\eta} d\Omega \\ &= \lim_{\eta \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Theta(\nu) e^{-i\nu t} d\nu \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - i\eta} d\omega \\ &= \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} \Theta(\Omega) e^{-i\Omega t - \eta t} d\Omega, \quad t > 0. \end{aligned} \quad (\text{B.19})$$

To express κ in terms of the spectral function we substitute with (B.19) in (B.10), so we get

$$\begin{aligned} \kappa &= \frac{i}{2VT} \int_0^{\infty} dt t \mathcal{G}(t) = \frac{i}{2VT} \int_0^{\infty} dt t \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} \Theta(\Omega) e^{-i\Omega t - \eta t} d\Omega \\ &= \frac{i}{2VT} \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} \Theta(\Omega) d\Omega \int_0^{\infty} dt t e^{-i\Omega t - \eta t}. \end{aligned} \quad (\text{B.20})$$

Taking into account that the spectral function is an odd function, $\Theta(\nu) = -\Theta(-\nu)$, we can see that only the imaginary part of $\int_0^{\infty} dt e^{-i\Omega t - \eta t}$ remains, what leads to

$$\kappa = -\frac{\pi}{2VT} \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} \Theta(\Omega) \frac{\partial \delta(\Omega)}{\partial \Omega} d\Omega = \frac{\pi}{2VT} \frac{\partial \Theta(\Omega)}{\partial \Omega} \Big|_{\Omega=0} \quad (\text{B.21})$$

or equivalently,

$$\kappa = \frac{\pi}{4VT} \lim_{\Omega \rightarrow 0} \frac{1}{\Omega} [\Theta(\Omega) - \Theta(-\Omega)]. \quad (\text{B.22})$$

From the representation (B.15) for the spectral function it can be shown that the thermal conductivity (B.22) can be expressed in terms of imaginary time Green's functions. Indeed, let us introduce the following thermal Green function

$$\Pi(\tau) = \text{Tr}\{\rho e^{H\tau} U_i(0) e^{-H\tau} U_i(0)\} \quad (\text{B.23})$$

and its Fourier transform

$$\Pi(i\omega_n) = \int_0^{\beta} \Pi(\tau) e^{i\omega_n \tau} d\tau, \quad \omega_n = \frac{2\pi n}{\beta}, \quad n = 0, 1, 2, \dots \quad (\text{B.24})$$

Inserting the complete set of energy eigenstates $|m\rangle, |n\rangle$ we can perform the integration over τ as indicated in (B.24), to find

$$\Pi(i\omega_k) = \frac{1}{Z} \sum_{n,m} e^{-\beta E_n} |\langle n | U_i | m \rangle|^2 \frac{e^{-(E_m - E_n)\beta} - 1}{E_n - E_m + i\omega_k}. \quad (\text{B.25})$$

If we now define an analytical function $\Pi(\omega)$ in such a way that at discrete points $\omega = i\omega_k$ it coincides with $\Pi(i\omega_k)$ and has a branch cut along the real axis, then the spectral function (B.15) is related to the discontinuity of $\Pi(\omega)$ across the cut

$$\Theta(\Omega) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0} [\Pi(\Omega + i\varepsilon) - \Pi(\Omega - i\varepsilon)]. \quad (\text{B.26})$$

Substituting (B.26) in (B.22) we arrive to equation (10) of Section 2.

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